NIDS Sesión 6

$\boxed{1}$ $\begin{cases} y'(t) = A(t)\, y(t) \\ y(t_0) = y_0 \end{cases}$ $\qquad A(t) = \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix}$

Compare

$y(t) = R(t, t_0)\, y_0$ with $e^{\int_{t_0}^{t} A(s)\, ds}\, y_0$

$$\begin{cases} R'(t, t_0) = A(t)\, R(t, t_0) \\ R(t_0, t_0) = I \end{cases}$$

$$R(t, t_0) = \begin{pmatrix} r_{11}(t) & r_{12}(t) \\ r_{21}(t) & r_{22}(t) \end{pmatrix}$$

$$\begin{pmatrix} r_{11}'(t) & r_{12}'(t) \\ r_{21}'(t) & r_{22}'(t) \end{pmatrix} = \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_{11}(t) & r_{12}(t) \\ r_{21}(t) & r_{22}(t) \end{pmatrix}$$

$$= \begin{pmatrix} t r_{11}(t) + r_{21}(t) & t r_{12}(t) + r_{22}(t) \\ 0 & 0 \end{pmatrix}$$

$(1) \begin{cases} r_{11}'(t) = t r_{11}(t) + r_{21}(t) \\ r_{11}(t_0) = 1 \end{cases}$ $\qquad (3) \begin{cases} r_{12}'(t) = t r_{12}(t) + r_{22}(t) \\ r_{12}(t_0) = 0 \end{cases}$

$(2) \begin{cases} r_{21}'(t) = 0 \\ r_{21}(t_0) = 0 \end{cases}$ $\qquad (4) \begin{cases} r_{22}'(t) = 0 \\ r_{22}(t_0) = 1 \end{cases}$

(2) $\quad x_{21}(t) = 0$

(4) $\quad x_{22}(t) = 1$

(1) $\begin{cases} x_{11}'(t) = t\, x_{11}(t) \\ x_{11}(t_0) = 1 \end{cases} \Rightarrow \quad x_{11}(t) = C e^{\frac{1}{2}t^2}$

$$x_{11}(t_0) = 1$$

$$\Rightarrow \quad x_{11}(t) = e^{\frac{1}{2}(t^2 - t_0^2)}$$

(3) $\quad x_{12}'(t) = \underbrace{t}_{g(t)} x_{12}(t) + 1 \qquad$ Non homogeneous

$$x_{12}(t) = x_{12}^{h}(t) + x_{12}^{p}(t)$$

$$x_{12}^{h}(t) = C e^{\frac{1}{2}t^2}$$

$$x_{12}^{p}(t) = \gamma(t)\, e^{G(t)}$$

$g(t) = t$

$G(t)$ primitive function of $g(t)$

(i.e. $\frac{1}{2}t^2$)

$$y'(t) = \underline{g(t)\, y(t)} + f(t)$$

$$y(t) = \gamma(t)\, e^{G(t)} \qquad y'(t) = \gamma'(t)\, e^{G(t)} + \underline{g(t)\, e^{G(t)}}$$

By substituting,

$$\gamma'(t)\, e^{G(t)} = f(t)$$

$$\gamma'(t) = f(t)\, e^{-G(t)}$$

$$\gamma(t) = \int f(s)\, e^{-G(s)}\, ds$$

in our case $f(t) = 7$

$$\gamma(t) = \int e^{-\frac{7}{2}s^2}\, ds$$

$$s_{12}^{p}(t) = \gamma(t) \underbrace{e^{G(t)}} = \int_{t_0}^{t} e^{\frac{7}{2}(t^2 - s^2)}\, ds$$

$$s_{12}(t_0) = \underbrace{s_{12}^{h}(t_0)}_{C e^{+\frac{7}{2}t_0^2}} + \underbrace{s_{12}^{p}(t_0)}_{0} = 0$$

$$\Rightarrow C = 0$$

Finally,

$$s_{12}(t) = s_{12}^{p}(t) = \int_{t_0}^{t} e^{\frac{7}{2}(t^2 - s^2)}\, ds$$

$$R(t, t_0) = \begin{pmatrix} e^{\frac{1}{2}(t^2 - t_0^2)} & \int_{t_0}^{t} e^{\frac{1}{2}(t^2 - s^2)} ds \\ 0 & 1 \end{pmatrix}$$

Compare with $e^{\int_{t_0}^{t} A(s)\, ds}$

Let's call $B(t) = \int_{t_0}^{t} A(s)\, ds$

$$= \int_{t_0}^{t} \begin{pmatrix} s & 1 \\ 0 & 0 \end{pmatrix} ds$$

$$= \begin{pmatrix} \frac{1}{2}(t^2 - t_0^2) & t - t_0 \\ 0 & 0 \end{pmatrix}$$

$$e^{B(t)} = \sum_{k=0}^{\infty} \frac{1}{k!} B(t)^k$$

$$B(t) = \begin{pmatrix} b_1 & b_2 \\ 0 & 0 \end{pmatrix} \qquad b_1 = \frac{1}{2}(t^2 - t_0^2) \qquad b_2 = t - t_0$$

$$B^2(k) = \begin{pmatrix} b_1 & b_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} b_1^2 & b_1 b_2 \\ 0 & 0 \end{pmatrix}$$

$$B(k)^k = \begin{pmatrix} b_1^k & b_1^{k-1} b_2 \\ 0 & 0 \end{pmatrix} \qquad k \geq 1$$

$$e^{B(k)} = \sum_{k=0}^{\infty} \frac{1}{k!} B^k(k)$$

$$= I + \sum_{k=1}^{\infty} \frac{1}{k!} B^k(k)$$

$$= \begin{pmatrix} 1 + \sum_{k=1}^{\infty} \frac{b_1^k}{k!} & \sum_{k=1}^{\infty} b_1^{k-1} b_2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{b_1} & \frac{b_2}{b_1} \sum_{k=1}^{\infty} b_1^{k} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{b_1} & \frac{b_2}{b_1}\left(e^{b_1} - 1\right) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{1}{2}(t^2 - t_0^2)} & \frac{2(t-t_0)}{(t^2-t_0^2)}\left(e^{\frac{1}{2}(t^2-t_0^2)} - 1\right) \\ 0 & 1 \end{pmatrix}$$

$$= e^{B(t)} = e^{\int_{t_0}^{t} A(s)\, ds}$$

Recall

$$\Phi(t, t_0) = \begin{pmatrix} e^{\frac{1}{2}(t^2 - t_0^2)} & \int_{t_0}^{t} e^{\frac{1}{2}(t^2-\tau^2)}\, d\tau \\ 0 & 1 \end{pmatrix}$$

Thus $e^{\int_{t_0}^{t} A(s)\, ds} \neq R(t, t_0)$

We would want that

$$\frac{d}{dt}\left(e^{\int_{t_0}^{t} A(s)\, ds} \right) = A(t)\, e^{\int_{t_0}^{t} A(s)\, ds}$$

But :

$$\frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{1}{k!} B(t)^k \right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d}{dt}\left(B(t)^k \right)$$

For a scalar function $f \in C^\infty(\mathbb{R}, \mathbb{R})$

$$\frac{d}{dt}\left(f^k(t) \right) = k\, \dot{f}(t)\, f(t)^{k-1}$$

$$= \frac{d}{dt}\Big(\underbrace{B(t) \cdots B(t)}_{k \text{ times}} \Big)$$

$$= \dot{B}(t) \underbrace{B(t) \cdots B(t)}_{k-1 \text{ times}} + \underline{B(t)}\, \dot{B}(t)\, \underline{B(t)} \cdots$$

$$+ \cdots$$

$$\frac{d}{dt}\left(B(t)^k \right)$$

$$= \dot{B}(t)\, B(t) \cdots B(t) + \cdots$$

$$+ B(t)\, \dot{B}(t)\, B(t) \cdots B(t) + \cdots$$

Condition: $B(t)$ and $\dot{B}(t) = A(t)$ must commute!

2) Consider the collocation method

$$\begin{cases} u(t_0) = y_0 \\ u'(t_0 + c_i h) = f(t_0 + c_i h, u(t_0 + c_i h)) \\ y_1 = u(t_0 + h) \end{cases}$$

This is equivalent to a RK method with coefficients

$$a_{ij} = \int_0^{c_i} l_j(\tau)\, d\tau \qquad b_i = \int_0^1 l_i(\tau)\, d\tau$$

Apply the same strategy as Esercizio 4 of Serie 2

Alternative

$$u(t) = \sum_{k=0}^{s} p_k (t - t_0)^k$$

From the initial condition

$$U(t_0) = g_0 = p_0 \in \mathbb{R}^m$$

For the other coefficients:

$$(1) \quad \sum_{k=1}^{s} k\, p_k (c_i h)^{k-1} = f\Big(t_0 + c_i h,\ g_0 + \sum_{k=1}^{s} p_k (c_i h)^k\Big)$$

$$i = 1, \dots, s$$

We would like to express it as the fixed-point equation $\underline{p} = F(\underline{p})$ (2)

$$\underline{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_s \end{pmatrix} \in \mathbb{R}^{sm} \qquad F : \mathbb{R}^{sm} \to \mathbb{R}^{sm}$$

The LHS of (1) is just $V\underline{p}$, where V is a Vandermonde matrix, which is invertible if the c_i are distinct

So, we can rewrite (1) as $Q1$ and then apply Banach's fixed point theorem.

$\boxed{3}$ $C(s-2)$ $\sum_{j=1}^{s} c_j^{k-1} a_{ij} = \frac{c_i^k}{k}$ $k = 1 \dots s-2$

$B(s-2)$ $\sum_{i=1}^{s} c_i^{k-1} b_i = \frac{1}{k}$ $k = 1 \dots s-2$

Rewrite $C(s-2)$ as

$$\underbrace{\begin{pmatrix} a_{11} & \dots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \dots & a_{ss} \end{pmatrix}}_{A \in \mathbb{R}^{s \times s}} \underbrace{\begin{pmatrix} c_1^0 & c_1^1 & \dots & c_1^{s-3} \\ c_2^0 & c_2^1 & & c_2^{s-3} \\ \vdots & \vdots & & \\ c_s^0 & c_s^1 & & c_s^{s-3} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} c_1 & \dots & \frac{c_1^{s-2}}{s-2} \\ c_2 & & \frac{c_2^{s-2}}{s-2} \\ & & \\ c_s & & \frac{c_s^{s-2}}{s-2} \end{pmatrix}}_{F}$$

$A = [\underline{a}_1 \dots \underline{a}_s]$

$C = \begin{pmatrix} c_1^T \\ \vdots \\ c_s^T \end{pmatrix} \in \mathbb{R}^{s \times (s-2)}$

$\underline{a}_j$ is the jth column of A

$$AC = \sum_{j=1}^{s} a_j c_j^T = F$$

Thus
$$\underbrace{\sum_{j=2}^{s} a_j c_j^T}_{\hat{A}\tilde{C}} = F - a_1 c_1^T - a_s c_s^T$$

$$\hat{A} = [\,\underline{a_2} \cdots \underline{a_{s-1}}\,] \in \mathbb{R}^{s\times(s-2)}$$

$$\tilde{C} = \begin{pmatrix} c_2^T \\ \vdots \\ c_{s-1}^T \end{pmatrix} \in \mathbb{R}^{(s-2)\times(s-2)}$$

$\tilde{C}$ is a Vandermonde matrix

Since the c_i are distinct,

$$\hat{A} = (F - a_1 c_1^T - a_s c_s^T)\,\tilde{C}^{-1}$$

For $B(s-2)$:

$$\sum_{i=1}^{s} b_i c_i^{k-1} = \frac{1}{k} \qquad k = 1 \dots s-2$$

Rewrite it as

$$\begin{pmatrix} b_1 & \cdots & b_s \end{pmatrix} C = \begin{pmatrix} 1 & 1/2 & \cdots & 1/(s-2) \end{pmatrix}$$

Same strategy

$\boxed{4}$ $C(s-2)$ holds $\Leftrightarrow$ the quadrature formula

$\{(c_j, \underbrace{a_{ij}}_{\text{quadrature weights}})\}_{j=1}^{s}$ integrates exactly polynomials of degree $s-3$ from 0 to c_i

quadrature points

i.e. :

$$\int_0^{c_i} p(x)\,dx = \sum_{j=1}^{s} a_{ij}\, p(c_j) \quad \text{holds for all } p \in \mathbb{P}^{s-3}$$

For $B(s-2)$, similarly,

$$\int_0^1 p(x)\,dx = \sum_{i=1}^{s} b_i\, p(c_i) \quad \text{holds for all } p \in \mathbb{P}^{s-3}$$

Consider the interpolating polynomial

$$l_j(\tau) = \prod_{k=2}^{s-1} \left(\frac{\tau - c_k}{c_j - c_k} \right) \in \mathbb{P}^{s-3}$$

$$\int_0^{c_i} l_j(\tau)\, d\tau = \sum_{k=1}^{s} a_{ik}\, l_j(c_k)$$

$$= \underbrace{a_{i1}}_{b_1} l_j(c_1) + \underbrace{\sum_{k=2}^{s-1} a_{ik}\, l_j(c_k)}_{a_{ij}} + \underbrace{a_{is}}_{0} l_j(c_s)$$

$(c_1 = 0,\ c_s = 1)$ $\quad$ $(l_j(c_k) = \delta_{jk})$

$$a_{ij} = \int_0^{c_i} l_j(\tau)\, d\tau - b_1\, l_j(0) \qquad \begin{array}{l} i = 1, \ldots, s \\ j = 2, \ldots, s-1 \end{array}$$

Similarly for b_i

$\boxed{5}$ / Alternative)

Let's define $\quad \tilde{p}_n(x) = \dfrac{p_n(x)}{x(1-x)} \in \mathbb{P}^{n-2}$

$$p_n(x) = \prod_{k=1}^{n}(x - c_k)$$

Let's use Euclidean division

for a polynomial $q(x) \in \mathbb{P}^{2n-3}$

$$q(x) = \underbrace{t(x)}_{\mathbb{P}^{n-1}} \underbrace{\tilde{p}_n(x)}_{\in \mathbb{P}^{n-2}} + \underbrace{r(x)}_{\mathbb{P}^{n-3}}$$

$$\int_0^1 q(x)\,dx = \underbrace{\int_0^1 \underbrace{t(x)\,\tilde{p}_n(x)}_{\in \mathbb{P}^{2n-3}}\,dx}_{\text{(I) integrate with Lobatto IIIA}} + \underbrace{\int_0^1 r(x)\,dx}_{\text{(II) integrate with Lobatto IIIB}^{(*)}}$$

(*) From exercise 4

Let $\hat{b}_i$ be the quadrature weights of Lobatto IIIA

(I) + (II) become

$$\sum_{i=1}^{n} \hat{b}_i\, t(c_i)\, \hat{p}_0(c_i) + \sum_{i=1}^{n} b_i\, r(c_i)$$

$$= \hat{b}_1\, t(c_1)\, \hat{p}_0(c_1) + \hat{b}_n\, t(c_n)\, \hat{p}_0(c_n) + \sum_{i=1}^{n} b_i\, r(c_i)$$

Since $\hat{b}_1 = b_1$, $\hat{b}_n = b_n$, we obtain

$$\sum_{i=1}^{n} b_i\, t(c_i)\, \tilde{p}_0(c_i) + \sum_{i=1}^{n} b_i\, r(c_i)$$

$$= \sum_{i=1}^{n} b_i \left(t(c_i)\, \tilde{p}_0(c_i) + r(c_i) \right)$$

$$= \sum_{i=1}^{n} b_i\, q(c_i)$$